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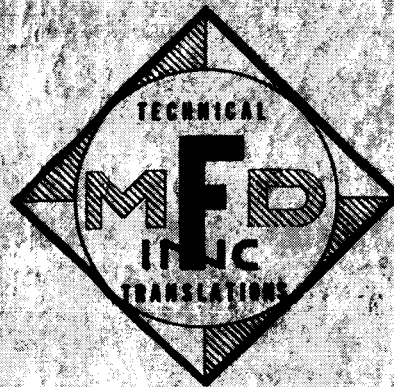
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MORRIS D. FRIEDMAN, INC.

On the Flow of a Conducting Fluid Around a Magnetized

Body

A. G. KULIKOVSKII

If the fluid conductivity is infinite, then the incident flow in which the magnetic field is assumed to be absent cannot penetrate the region occupied by the magnetic field (the magnetic lines of force are "frozen" into the fluid for infinite conductivity). Hence, it follows that the region being streamlined consists of the magnetized body and a "cavern", a region in which is the magnetic field. This region can be empty or can be filled with fluid. To be specific, let us consider the cavern to be filled with the streamlining fluid since the fluid, although slowly, can penetrate within the cavern for a large but finite conductivity.

Let us consider the stationary streamline flow case. Generally speaking, steady fluid motion is possible within the cavern but we shall subsequently consider that the fluid is at rest within the cavern and we shall seek the shape of the cavern and the magnetic field therein. Let us note that the velocity in the cavern certainly equals zero when all the magnetic lines of force start and terminate in the body. Putting $v = 0$ in the magnetohydrodynamic equations, we obtain:

(1)
$$\text{div } \vec{H} = 0 \; ; \quad \text{grad } p_{cav} = \frac{1}{4\pi}[\text{rot } \vec{H} \cdot \vec{H}]$$

where p_{cav} is the pressure within the cavern, where the following conditions must be satisfied on the cavern-flow boundary (p_f is pressure in the flow; $\vec{n}$ is the unit normal vector)

(2)
$$(\vec{H} \cdot \vec{n}) = 0 \; ; \quad p_f = p_{cav} + \frac{1}{8\pi} H^2$$

The first condition results from the absence of magnetic field sources and is satisfied on the boundary of the region being streamlined and the second expresses the fact that the normal stresses [the tangential magnetic stress automatically equals zero by compliance with the first condition of (2)] are equal.

A magnetic field can be assigned within the body (when the body conductivity is infinite, say) or the currents

(3)
$$\vec{J}_b = \frac{c}{4\pi} \text{ rot } \vec{H}$$

can be given in the body but the magnetic field intensity in the body must be determined from the solution of the problem. Hence, the normal components of

of the vectors $\vec{H}$ and rot $\vec{H}$ must be continuous along the normal at the cavern-body interface.

But equations (1) with their boundary conditions and conditions within the body do not give a unique solution of the problem. Actually, the stationary motion being considered as arising from a rest state can depend on how the magnetic field was "frozen" in the fluid in the original state.

In order that the solution of the problem should be independent of the original state and should only depend on the conditions within the body, let us impose the following additional conditions (which are not obligatory, generally) on the magnetic field.

Let

(4)   rot rot $\vec{H}$ = 0

within the cavern and let

(5) $[\text{rot } \vec{H} \cdot \vec{n}] = 0$

on the cavern-stream interface. These conditions were obtained in an attempt to isolate that solution to which stationary solutions with finite, constant conductivity σ approach as $\sigma \to \infty$. Here, it was assumed that there is a region where $\vec{v} \to 0$ in the neighborhood of the body and also that the last term in a generalized Ohm's law

$$\vec{j} = \sigma \left\{ \vec{E} + \frac{1}{c} [\vec{v} \cdot \vec{H}] \right\}$$

approaches zero as $\sigma \to \infty$.

Equations (3) and (4) can be combined into one equation describing the magnetic field in the whole region being streamlined:

(6)   rot rot $\vec{H} = \frac{4\pi}{c}$ rot $\vec{j}_b$

where $\vec{j}_b$ is a function agreeing with the currents within the body and zero outside, where the vector $\vec{j}_b$ itself must be given on the body-flow interface, if such exists.

Now, let us show that if the cavern shape is known beforehand and the region being streamlined is finite, then (6) with the first equation of (1) and their boundary conditions can have only an unique solution. To do this, let us show that the difference of the two solutions $\vec{H}_1 - \vec{H}_2 = \vec{H}_3$ is zero. Actually, $\vec{H}_3$ satisfies (4) everywhere in the region being streamlined and satisfies (5) and the first condition of (2) on its whole boundary. Hence, we conclude that

(7)   rot $\vec{H}_3$ = grad f

and that $f = $ const on the boundary. Taking the div of both sides of (7), we obtain $\Delta f = 0$ and, therefore, grad $f = 0$ everywhere in the region being streamlined. Then, it follows from (7) that $\vec{H}_3 = $ grad ψ and, taking the first equality of (1) into account, we obtain $\Delta \psi = 0$ and the equality $\frac{\partial \psi}{\partial n} = 0$ will be satisfied on the boundary because of the first condition of (2). Therefore, $\psi = $ const and $\vec{H}_3 = 0$.

Equation (6) is simplified when the problem is planar or axisymmetric. Let us introduce the x,y,z coordinates in the planar case (the x and y axes are in the plane of symmetry) and the coordinates x,y,ϕ in the axisymmetric case (the x axis is the axis of symmetry). The currents within the cavern $\vec{j} = \frac{c}{4\pi}$ rot $\vec{H}$ are in the xy plane in both cases. This follows from (4) and (5) as well as from $\vec{H} = 0$ in the incident flow. Consequently, (6) can be written as follows:

In the planar case

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{4\pi}{c} j_{bz}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \frac{4\pi}{c}\left(\frac{\partial j_{bx}}{\partial y} - \frac{\partial j_{by}}{\partial x}\right)$$

In the axisymmetric case

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{4\pi}{c} j_{b\phi}$$

$$\frac{\partial^2 H_\phi}{\partial x^2} + \frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(yH_\phi)\right] = \frac{4\pi}{c}\left(\frac{\partial j_{bx}}{\partial y} - \frac{\partial j_{by}}{\partial x}\right)$$

where condition (5) will be

In the planar case $\quad \dfrac{\partial H_z}{\partial n} = 0$

In the axisymmetric case $\quad \dfrac{\partial H_\phi}{\partial n} + \dfrac{H_\phi}{y} n_y = 0$

where $\hat{n} = \vec{i}n_x + \vec{j}n_y$ is the unit normal vector. If $\dfrac{\partial j_{bx}}{\partial y} - \dfrac{\partial j_{by}}{\partial x} = 0$, then H_x and H_y can equal zero and the problem reduces to the integration of the following two equations in this case:

$$\text{rot } \vec{H} = \vec{j}_b \quad ; \qquad \text{div } \vec{H} = 0$$

and we obtain $p_{cav} = $ const from the second equation of (1).

Let us analyze several very simple examples assuming that $H_z = 0$, $H_\phi = 0$.

1. Flow around a plane magnetic dipole perpendicular to the flow by an

incompressible fluid. The surface being streamlined is a cylinder whose radius we will denote by a .. The magnetic field within the cylinder and the fluid velocity outside it are given by

$$H = \text{grad } H_{oy}\left(1 - \frac{a^2}{x^2 + y^2}\right) \quad ; \quad v = \text{grad } U_{ox}\left(1 + \frac{a^2}{x^2 + y^2}\right)$$

where U_o and H_o are constants related by means of

$$H_o^2 = 8\pi\rho U_o^2$$

The pressure within the cylinder p_c is constant and related to the stagnation pressure of the incident flow p_o as follows:

$$p_o - p_c = \frac{1}{4\pi} H_o^2$$

2. Supersonic flow around a wedge along whose surface a current of constant density i flows parallel to the edge of the wedge. To be specific, let us consider that the current flows in opposite directions on the different edges of the wedge. The flow picture and the behavior of the magnetic lines of force are shown on figure 1.
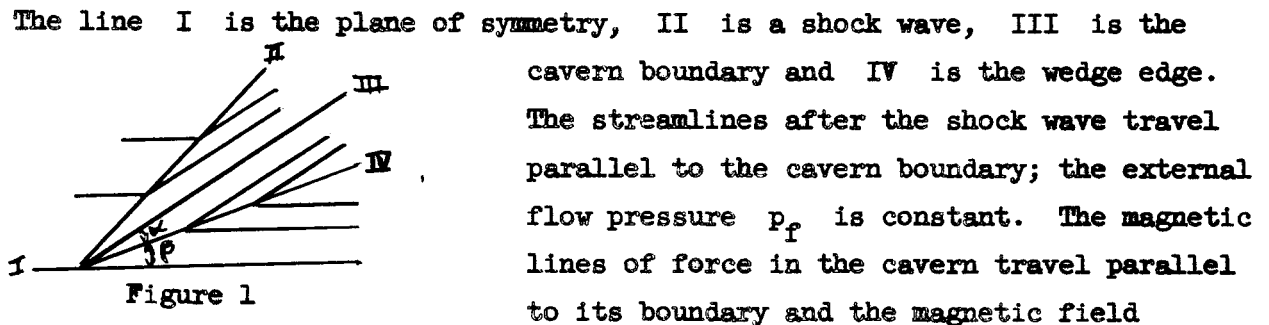
The line I is the plane of symmetry, II is a shock wave, III is the

Figure 1

cavern boundary and IV is the wedge edge. The streamlines after the shock wave travel parallel to the cavern boundary; the external flow pressure p_f is constant. The magnetic lines of force in the cavern travel parallel to its boundary and the magnetic field intensity H_{cav} is constant. The pressure in the cavern is related to the flow pressure by means of the relation

$$p_f = p_{cav} + \frac{1}{8\pi} H_{cav}^2$$

The magnetic field intensity within the wedge H_w is constant and related to H_{cav} and i by means of

$$H_{cav}\sin \alpha = H_w \sin \beta \quad ; \quad i = \frac{c}{4\pi}(H_{cav}\cos \alpha + H_w \cos\beta)$$

It is easy to see that the minimum current intensity required to maintain the pressure difference $p_f - p_{cav}$ is reached at $\alpha = 0$ and equals

$$i_{min} = \frac{c}{\sqrt{2\pi}} \sqrt{p_f - p_{cav}}$$

where i_{min} is a very large quantity. Thus, for example, if $p_f - p_{cav} = 1$ atm ,

then $i_{min} = 4000$ $^a/cm$.

Thus, a necessary condition for the existence of a cavern is $i > i_{min}$. Otherwise, the cavern is missing and the usual flow around a wedge occurs.

3. Supersonic flow around a cone along whose surface flows a current of constant density i directed perpendicularly to the cone generators. The streamline picture remains approximately the same as the streamlines behind the shock wave and the magnetic lines of force in the cavern cease to be straight. The behavior of the magnetic lines of force in the cavern is given by

$$H_r = C_1(1 + \cos\theta \ln\tan\tfrac{1}{2}\theta) + C_2\cos\theta$$
$$H_\theta = C_1(\cot\theta - \sin\theta \ln\tan\tfrac{1}{2}\theta) - C_1\sin\theta$$

Here H_r is the magnetic field intensity component along the radius-vector drawn from the cone vertex; θ is the angle made by the radius-vector with the axis of symmetry; H_θ is the magnetic field intensity component perpendicular to H_r . The constants C_1 and C_2 are determined from the conditions on the cavern boundary (for $\theta = \alpha + \beta$)

$$H_\theta = 0 \ ; \quad p_f = p_{cav} + \frac{1}{8\pi} H_r^2$$

The magnetic field intensity is constant within the cone and the magnetic lines of force are parallel to the axis of symmetry. The surface current density is determined from the equality

$$i = \frac{c}{4\pi}(H_{cav\ t} - H_{c\ t})$$

where $H_{cav\ t}$ and $H_{c\ t}$ are the tangential magnetic field intensities of the cavern and the cone, respectively, on the cone surface. An i_{min} also exists for a cone, which is expressed by the same formula as for the wedge.

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